

THE AUTOHOMEOMORPHISM GROUP OF THE ČECH-STONE COMPACTIFICATION OF THE INTEGERS

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ABSTRACT. It is shown to be consistent that there is a nontrivial autohomeomorphism of $\beta\mathbb{N} \setminus \mathbb{N}$, yet all such autohomeomorphisms are trivial on a dense P -ideal. Furthermore, the cardinality of the autohomeomorphism group of $\beta\mathbb{N} \setminus \mathbb{N}$ can be any regular cardinal between 2^{\aleph_0} and $2^{2^{\aleph_0}}$. The model used is one due to Velickovic in which, coincidentally, Martin's Axiom also holds.

1. INTRODUCTION

An automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ —or, equivalently, an autohomeomorphism of $\beta\mathbb{N} \setminus \mathbb{N}$ —is said to be *trivial* if there is a bijection between co-finite subsets of the integers which induces it; an automorphism is said to be *somewhere trivial* if its restriction to $\mathcal{P}(A)/[A]^{<\aleph_0}$ is trivial for some $A \in [\omega]^{\aleph_0}$. It was shown by Shelah, [3, pp. 129–152] that it is consistent with ZFC that all automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ are trivial; therefore, it is also consistent that the size of the autohomeomorphism group of $\beta\mathbb{N} \setminus \mathbb{N}$ is 2^{\aleph_0} . This is in contrast to most other models of set theory where the size of the autohomeomorphism group of $\beta\mathbb{N} \setminus \mathbb{N}$ is $2^{2^{\aleph_0}}$. This might lead one to ask whether there is a dichotomy similar to that for Borel sets, that is, the size of the autohomeomorphism group of $\beta\mathbb{N} \setminus \mathbb{N}$ is either 2^{\aleph_0} or $2^{2^{\aleph_0}}$. The present paper will show that no such dichotomy exists, as well as providing some detailed information about the nature of automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ in a certain class of models. In particular, it will be shown that in these models all automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ are trivial on a dense P -ideal. The fact that they are trivial on a dense set was established in [5].

In order to be more precise the following definitions will be introduced.

Definition 1.1. The relation \equiv^* has the standard meaning; namely, $A \equiv^* B$ if and only if $|A \Delta B| < \aleph_0$ (here, $A \Delta B = (A \setminus B) \cup (B \setminus A)$). Also, $A \subseteq^* B$ is defined to mean that $|A \setminus B| < \aleph_0$. If $A \subseteq \omega$, then the equivalence class of A with respect to \equiv^* will be denoted by $[A]$.

The notion of triviality can now be precisely formulated.

Definition 1.2. A homomorphism $\Phi : \mathcal{P}(\omega)/[\omega]^{<\aleph_0} \rightarrow \mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ is said to be trivial on $A \subseteq \omega$ if there is $A' \equiv^* A$ and a one-to-one function $f : A' \rightarrow \omega$ such that $\Phi([B]) = [f(B)]$ for every $B \subseteq A$. The function f is said to induce Φ on A .

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A homomorphism will be said to be somewhere trivial if there is some $A \in [\omega]^{\aleph_0}$ on which it is trivial. A homomorphism is trivial if it is trivial on ω .

It has already been mentioned that it was shown in [3] that it is consistent that all automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ are trivial. The argument relied on the oracle chain condition, and it was not clear what the effect of Martin's Axiom was on the question. This was partially answered in [4] where it was shown that the Proper Forcing Axiom (PFA) implies that all automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ are trivial; for related results see [2]. The other half of the answer was provided by Velickovic in [9] where it is shown that it is consistent with Martin's Axiom that a nontrivial automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ exists.

2. NOTATION AND BACKGROUND

The notation used throughout will adhere to accepted convention. If n and m are integers, then $[n, m)$ will denote the interval $m \setminus n$ even though the latter notation is more compact. When a partial order appears in a forcing context, greater conditions will be assumed to force more information unless a natural ordering, such as \subseteq or \subseteq^* , on the set is used.

Notation 2.1. The notation i_X will be used to denote the constant function whose domain is X and which has value i at each point in X .

Notation 2.2. Whenever reference is made to a topology on $\mathcal{P}(\omega)$, this will be to the Cantor set topology under the canonical identification of 2^ω with $\mathcal{P}(\omega)$; in other words, a natural base for this topology consists of all sets of the form

$$\{A \subseteq \omega : 1_A \cup 0_{\omega \setminus A} \supseteq g\}$$

where g is a finite partial function from ω to 2.

Notation 2.3. The notation $\overset{\circ}{x}$ will be used to indicate names not of the form \check{x} in forcing statements, the assumption being that all symbols occurring in a forcing expression are of the form \check{x} unless explicitly indicated otherwise by \circ . For example, in the expression $p \Vdash \text{"}n \in \overset{\circ}{y}\text{"}$ it is being assumed that n is the same as \check{n} .

Definition 2.1. If Φ is an automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$, define $\mathcal{T}(\Phi)$ to be the ideal of all $X \in [\omega]^{\aleph_0}$ such that Φ is trivial on X .

The following theorem of [9] offers an alternate characterization of triviality which has proven to be very useful.

Lemma 2.1 (Velickovic). *If $\Phi : \mathcal{P}(\omega)/[\omega]^{<\aleph_0} \rightarrow \mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ is an automorphism and there exist Borel functions $\psi_n : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ for $n \in \omega$ such that for all but a meagre set of $A \in \mathcal{P}(\omega)$ there is $n \in \omega$ such that $[\psi_n(A)] = \Phi([A])$, then Φ is trivial.*

The meagre set of exceptions is not mentioned in [9], but it follows from minor modifications to the argument there.

3. AUTOMORPHISMS OF $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ ARE TRIVIAL ON A DENSE P -IDEAL

The arguments to be presented in this section will be a modification and combination of arguments from [3, pp. 129–152] (or [6, pp. 175–193]), [4] and [9]. The

following partial order \mathbb{P} , was introduced by Velickovic in [9], following Baumgartner, to add a nontrivial automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ while doing as little else as possible—at least assuming PFA.

Definition 3.1. The partial order \mathbb{P} is defined to consist of all bijections $f : A \rightarrow B$ where

- $A \subseteq \omega$ and $B \subseteq \omega$;
- for all $i \in A$ and $n \in \omega$, $2^n \leq f(i) < 2^{n+1}$ if and only if $2^n \leq i < 2^{n+1}$;
- $\limsup_{n \rightarrow \omega} |[2^n, 2^{n+1}) \setminus A| = \omega$ and so, by the previous condition, $\limsup_{n \rightarrow \omega} |[2^n, 2^{n+1}) \setminus B| = \omega$.

The ordering on \mathbb{P} is \subseteq^* .

The terms 2^n are not crucial in the definition of \mathbb{P} , since any sequence of intervals whose size tends to infinity could equally well have been used. Further modifications to the partial order are also possible (some can be found in [4]) but will not be important in the present context. It is, however, useful to note the following observation from [5].

Lemma 3.1. Assume MA_η and that $\{f_\xi\}_{\xi \in \eta}$ is an increasing sequence from \mathbb{P} . Suppose further that there is f' such that $f' \supseteq^* f_\xi$ for each $\xi \in \eta$. Then there is $f \in \mathbb{P}$ such that $f \supseteq^* f_\xi$ for each $\xi \in \eta$.

Definition 3.2. The countable support product of κ copies of \mathbb{P} will be denoted by $\mathbb{P}(\kappa)$; in other words, if $f \in \mathbb{P}(\kappa)$, then $f : \kappa \rightarrow \mathbb{P}$ and $f(\alpha) = \emptyset$ for all but countably many α . The operations and relations $\cup, \cap, \setminus, \subseteq, \supseteq, \subseteq^*$ and \supseteq^* on elements of $\mathbb{P}(\kappa)$ will refer to the coordinatewise operations and relations; so, for example, $(f \setminus g)(\alpha) = f(\alpha) \setminus g(\alpha)$. The ordering on $\mathbb{P}(\kappa)$ is \subseteq^* in the sense just defined; in other words, $f \subseteq^* f'$ if and only if $f(\alpha) \subseteq^* f'(\alpha)$ for each $\alpha \in \kappa$.

The relation \subseteq^* on $\mathbb{P}(\kappa)$ is not actually an ordering but only a pre-ordering. However, it can easily be made into an ordering by identifying f with g if $f \subseteq^* g \subseteq^* f$. If κ is finite, then the resulting equivalence classes are countable, but otherwise they are uncountable. However, to avoid having to deal with equivalence classes, the pre-ordering on $\mathbb{P}(\kappa)$ will be used in the sequel.

Lemma 3.2. $\mathbb{P}(\kappa)$ is countably closed.

Proof. Given a sequence $\{f_n\}_{n \in \omega} \subseteq \mathbb{P}(\kappa)$ such that $f_n \subseteq^* f_{n+1}$ for each $n \in \omega$, choose inductively $k_n(\alpha)$ for $\alpha \in \kappa$ such that

$$f_\omega(\alpha) = \bigcup_{n \in \omega} f_n(\alpha) \upharpoonright (\omega \setminus k_n(\alpha))$$

is a function for each $\alpha \in \kappa$. Now apply Lemma 3.1 to each coordinate. □

Definition 3.3. For any G that is a centred subset of $\mathbb{P}(\kappa)$, define $\Phi_G^\alpha : \mathcal{P}(\omega)/[\omega]^{<\aleph_0} \rightarrow \mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ by

$$\Phi_G^\alpha([X]) = \begin{cases} [\{g(\alpha)(i) : i \in X\}] & \text{if } (\exists g \in G)(X \subseteq \text{domain}(g(\alpha))), \\ [\omega \setminus \{g(\alpha)(i) : i \in \omega \setminus X\}] & \text{if } (\exists g \in G)(\omega \setminus X \subseteq \text{domain}(g(\alpha))), \end{cases}$$

and, in the case that $\kappa = 1$, the superscript will be omitted.

Genericity easily yields the following.

Lemma 3.3. *If $G \subseteq \mathbb{P}(\kappa)$ is a generic filter on $\mathbb{P}(\kappa)$, then Φ_G^α is a nontrivial automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ and $\Phi_G^\alpha \neq \Phi_G^\beta$ unless $\alpha = \beta$.*

An important fact is the result of Velickovic [9] that if F is \mathbb{P} -generic over a model V of PFA, then in $V[F]$, not only is there a nontrivial autohomeomorphism of $\beta\mathbb{N} \setminus \mathbb{N}$, but MA also holds. Indeed, no new subsets of the ground model of size \aleph_1 are added. It will be shown that a closer analysis of these models yields that there are few automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ in them.

Notation 3.1. The notation \mathbb{S} will be used to denote Silver forcing, which consists of partial functions from ω to 2 with co-infinite domain.

Definition 3.4. Given $\mathfrak{S} \subseteq \mathbb{S}$ define $\mathbb{S}(\mathfrak{S}) \subseteq \mathbb{S}$ to be the partial order consisting of all $s \in \mathbb{S}$ such that there is some $s' \in \mathfrak{S}$ such that $s \subseteq^* s'$. The ordering is defined to be \subseteq .

Definition 3.5. For f and g in $\mathbb{P}(\kappa)$ define $f \leq_\kappa^* g$ if and only if $f \subseteq^* g$ and $f(\alpha) \subseteq g(\alpha)$ for all but finitely many $\alpha \in \kappa$.

From Lemma 3.1 it follows that, given a sequence $\{f_\xi\}_{\xi \in \omega_1} \subseteq \mathbb{P}(\kappa)$, it will be useful to find an element $f \in \mathbb{P}$ such that $f_\xi \subseteq^* f$ for each $\xi \in \omega_1$. The following definition of a partial order is designed to do precisely this.

Definition 3.6. Given $\mathfrak{F} \subseteq \mathbb{P}(\kappa)$ define $\mathbb{P}(\mathfrak{F}) \subseteq \mathbb{P}(\kappa)$ to be the partial order consisting of all $f \in \mathbb{P}(\kappa)$ such that there is some $f' \in \mathfrak{F}$ such that $f \leq_\kappa^* f'$. The ordering on $\mathbb{P}(\mathfrak{F})$ is coordinatewise inclusion. Given $f \in \mathbb{P}(\kappa)$, $A \subseteq \kappa$ and an integer M , let $f \langle A, M \rangle$ be the function defined by

$$f \langle A, M \rangle(\alpha) = \begin{cases} f(\alpha) \upharpoonright 2^M & \text{if } \alpha \in A, \\ \emptyset & \text{if } \alpha \in \kappa \setminus A. \end{cases}$$

Define $f[A, M] = f \setminus f \langle A, M \rangle$ and let $\mathbb{P} \langle A, M \rangle = \{f \in \mathbb{P}(\kappa) : f \langle A, M \rangle = f\}$.

For the purposes of this paper, $\mathbb{P}(\mathfrak{F})$ and $\mathbb{S}(\mathfrak{S})$ can be thought of as amoeba forcings for $\mathbb{P}(\kappa)$ and \mathbb{S} in the sense that the generic objects they produce yield elements of $\mathbb{P}(\kappa)$ and \mathbb{S} , respectively. The following two technical results contain the details of the fusion constructions on the amoeba partial orders that will be required in later arguments.

Lemma 3.4. *Given a \leq_κ^* -directed family $\mathfrak{F} \in [\mathbb{P}(\kappa)]^{\aleph_0}$ and a countable partial order \mathbb{Q} , both belonging to a countable model \mathfrak{M} of some large fragment of set theory, there is an $f \in \mathbb{P}(\kappa)$ such that*

- (1) $f' \leq_\kappa^* f$ for all $f' \in \mathfrak{F}$;
- (2) for each $D \in \mathfrak{M}$ that is dense in $\mathbb{P}(\mathfrak{F}) \times \mathbb{Q}$, for each $q \in \mathbb{Q}$ and each finite set $A \subseteq \kappa \cap \mathfrak{M}$, there are arbitrarily large integers M such that, given $h \in \mathbb{P} \langle A, M \rangle$, there is some $(p, \bar{q}) \in D$ such that $h \subseteq p$, $q \leq \bar{q}$ and $p[A, M] \subseteq f$.

Proof. The argument here is standard. Let $\{f_n\}_{n \in \omega}$ be a cofinal \leq_κ^* -increasing sequence in $\mathbb{P}(\mathfrak{F})$. Let \mathfrak{D} be the collection of all dense subsets of $\mathbb{P}(\mathfrak{F}) \times \mathbb{Q}$ belonging to \mathfrak{M} , let $\{(D_i, q_i, A_i)\}_{i \in \omega}$ enumerate infinitely often $\mathfrak{D} \times \mathbb{Q} \times [\kappa \cap \mathfrak{M}]^{<\aleph_0}$, and choose

inductively a sequence of conditions $(p_n, \bar{q}_n) \in \mathfrak{M} \cap (\mathbb{P}(\mathfrak{F}) \times \mathbb{Q})$, finite sets A_n^* and integers M_n such that

- $$\begin{aligned}
 (3.1) \quad & A_n^* \cup A_{n+1} \subseteq A_{n+1}^*, \\
 (3.2) \quad & (\forall h \in \mathbb{P}\langle A_n^*, M_n \rangle)(h \cup p_n[A_n, M_n], \bar{q}_n) \in D_n, \\
 (3.3) \quad & (\forall \beta \in A_n)(\exists j < M_n)[2^j, 2^{j+1}) \setminus \text{domain}(p_n(\beta))| > n, \\
 (3.4) \quad & p_{n+1}\langle A_n^*, M_n \rangle = p_n\langle A_n^*, M_n \rangle, \\
 (3.5) \quad & p_n \subseteq p_{n+1}, \\
 (3.6) \quad & q_n \leq \bar{q}_n, \\
 (3.7) \quad & f_n \leq_\kappa^* p_n.
 \end{aligned}$$

Let $f = \bigcup_{n \in \omega} p_n$. It follows from Conditions (3.3), (3.4), (3.1) and (3.5) that $f \in \mathbb{P}(\kappa)$. From Condition (3.7), it follows that $f \supseteq p_n \geq_\kappa^* f_n$, and so Condition (1) is satisfied. To see that Condition (2) is satisfied use Conditions (3.2).

To see that the induction can be carried out, suppose that p_n has been constructed. Let $A_{n+1}^* = A_{n+1} \cup A_n^*$. Choose M_{n+1} so large that

$$(\forall \beta \in A_{n+1}^*)(\exists j < M_{n+1})|[2^j, 2^{j+1}) \setminus \text{domain}(p_n(\beta))| > n + 1$$

and let $\{h_i\}_{i=0}^k$ enumerate $\mathbb{P}\langle A_{n+1}^*, M_{n+1} \rangle$. Using the density of D_{n+1} in $\mathbb{P}(\mathfrak{F}) \times \mathbb{Q}$, find a sequence of conditions $\{(p_i^*, q_i^*)\}_{i=0}^k$ such that

- $$\begin{aligned}
 (3.8) \quad & p_{i+1}^*[A_{n+1}^*, M_{n+1}] \supseteq p_i^*[A_{n+1}^*, M_{n+1}], \\
 (3.9) \quad & p_i^* \in D_{n+1}, \\
 (3.10) \quad & p_i^*\langle A_{n+1}^*, M_{n+1} \rangle = h_i, \\
 (3.11) \quad & q_i^* \leq q_{i+1}^*
 \end{aligned}$$

for each $i \leq k$. Let $\bar{q}_{n+1} = q_k^*$. Using the directedness of \mathfrak{F} it is easy to find $p_{n+1} \supseteq p_k^*$ such that Condition (3.7) is satisfied and, moreover, $p_{n+1}\langle A_{n+1}^*, M_{n+1} \rangle = p_n\langle A_{n+1}^*, M_{n+1} \rangle$. This last requirement guarantees that Condition (3.3) is satisfied by the choice of M_{n+1} . The fact that $p_k^*[A_{n+1}^*, M_{n+1}] \subseteq p_{n+1}[A_{n+1}^*, M_{n+1}]$ guarantees that Conditions (3.2) and (3.5) are also satisfied. \square

Corollary 3.1. *Given a \leq_κ^* -directed family $\mathfrak{F} \in [\mathbb{P}(\kappa)]^{\aleph_0}$ and a countable partial order \mathbb{Q} , both belonging to a countable model \mathfrak{M} of some large fragment of set theory, there is an $f \in \mathbb{P}(\kappa)$ such that $f' \leq_\kappa^* f$ for all $f' \in \mathfrak{F}$ and, moreover, each $D \in \mathfrak{M}$ that is dense in $\mathbb{P}(\mathfrak{F}) \times \mathbb{Q}$ remains dense in $\mathbb{P}(\{f\}) \times \mathbb{Q}$ for any $\bar{f} \in \mathbb{P}(\kappa)$ such that $f \leq_\kappa^* \bar{f}$.*

Proof. Let f be the element of $\mathbb{P}(\kappa)$ whose existence follows from Lemma 3.1, and suppose that $\bar{f} \in \mathbb{P}(\kappa)$ is such that $f \leq_\kappa^* \bar{f}$. If $D \in \mathfrak{M}$ is dense in $\mathbb{P}(\mathfrak{F}) \times \mathbb{Q}$ and $(f', q) \in \mathbb{P}(\{f\}) \times \mathbb{Q}$, choose $A \in [\kappa]^{<\aleph_0}$ and m so large that $f[A, m] \subseteq \bar{f}[A, m]$ and $f'[A, m] \subseteq \bar{f}[A, m]$. Then, let $M \geq m$ be such that for each $h \in \mathbb{P}\langle A, M \rangle$ there is some $(p, \bar{q}) \in D$ such that $h \subseteq p$, $p \setminus h \subseteq f$ and $\bar{q} \geq q$. Then there is $(p, \bar{q}) \in D$ such that $f'\langle A, M \rangle \subseteq p$, $p[A, M] \subseteq f[A, M] \subseteq \bar{f}[A, M]$ and $\bar{q} \geq q$. Since $f'[A, M] \subseteq \bar{f}[A, M]$, it follows that (p, \bar{q}) is compatible with (f', q) . \square

Corollary 3.2. *Given $\mathfrak{F} \in [\mathbb{P}(\kappa)]^{\aleph_0}$ and a countable partial order \mathbb{Q} , both belonging to a countable model \mathfrak{M} of some large fragment of set theory, there is $f \in \mathbb{P}(\kappa)$*

satisfying the conclusions of Lemma 3.4 and, moreover, for each $\mathbb{P}(\mathfrak{F})$ -name, $\overset{\circ}{g}$,¹ for an element of ${}^\omega\omega$ there is some $d \in {}^\omega\omega$ such that $1 \Vdash_{\mathbb{P}(\{\mathfrak{F}\})} \overset{\circ}{g} \leq^* d$. Moreover, if $f \leq_\kappa^* \bar{f}$, it is still true that $1 \Vdash_{\mathbb{P}(\{\bar{f}\})} \overset{\circ}{g} \leq^* d$.

Proof. The conclusion of Lemma 3.4 immediately yields a dominating function d from f . To see this, let D_n be the set of conditions deciding the value of $\overset{\circ}{g}(n)$, let A_n and M_n be such that, given $h \in \mathbb{P}\langle A_n, M_n \rangle$, there is some $p_h \in D_n$ such that $h \subseteq p_h$, $q \leq 1_{\mathbb{Q}}$ and $p_h[A, M] \subseteq f$, and let $d(n, h)$ be the value of $\overset{\circ}{g}(n)$ decided by p_h . Let

$$d(n) = \max_{h \in \mathbb{P}\langle A_n, M_n \rangle} d(n, h) + 1$$

and observe that the role of \mathbb{Q} is irrelevant. The final sentence follows from Corollary 3.1. \square

Lemma 3.5. *Given $\mathfrak{S} \in [\mathbb{S}]^{\aleph_0}$ and a countable partial order \mathbb{Q} , both belonging to a countable model \mathfrak{M} of some large fragment of set theory, and $U \subseteq \omega$ such that $U \cap \text{domain}(s)$ is finite for each $s \in \mathfrak{S}$, there is a comeagre set of partitions $U = U^0 \cup U^1$ such that each $D \in \mathfrak{M}$ that is dense in $\mathbb{Q} \times \mathbb{S}(\mathfrak{S})$ remains predense in $\mathbb{Q} \times \mathbb{S}(\mathfrak{S} \cup \{0_{W \setminus Y} \cup 1_Y\})$ for any choice of $a \in 2$ and $Y \subseteq W \subseteq U^a$.*

Proof. This is the argument used to establish (*1) on page 175 of [3]. \square

Notation 3.2. For the rest of this section, if $\Phi : \mathcal{P}(\omega)/[\omega]^{<\aleph_0} \rightarrow \mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ is an automorphism, then Φ^* will denote an arbitrarily chosen lifting of Φ . In other words, $\Phi^* : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ is such that $\Phi^*(A) \in \Phi([A])$ for each $A \subseteq \omega$. This convention will be adhered to, even if Φ is only a forcing name for an automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$.

Lemma 3.6. *Suppose that Φ is an automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ in V . Let $\{A_n\}_{n \in \omega}$ and A be subsets of the integers such that $A \supseteq^* A_n$ for each n and there do not exist Borel functions $\{\psi_n : n \in \omega\}$ such that for all but a meagre set of $C \in \mathcal{P}(A) \cap V$ there is some $n \in \omega$ such that $\Phi^*(C) \triangle \psi_n(C) \subseteq \bigcup_{i \in k} \Phi^*(A_i)$ for some $k \in \omega$. If V' is obtained by adding a Cohen real to V , then, in V' , there also do not exist Borel functions $\{\psi_n : n \in \omega\}$ such that for all but a meagre set of $C \in \mathcal{P}(A) \cap V$ there is some $n \in \omega$ such that $\Phi^*(C) \triangle \psi_n(C) \subseteq \bigcup_{i \in k} \Phi^*(A_i)$ for some $k \in \omega$.*

Proof. Suppose that V' is obtained by forcing with the countable partial order \mathbb{C} and that ψ_n are \mathbb{C} -names for Borel functions such that for each $C \in \mathcal{P}(A) \cap V$ there is some $n \in \omega$ such that $\Phi^*(C) \triangle \psi_n(C) \subseteq \bigcup_{i \in k} \Phi^*(A_i)$ for some $k \in \omega$. For each integer n , let G_n be a co-meagre set such that $\psi_n \restriction G_n$ is continuous. Define

$$\psi_n^p = \{(A, B) : p \Vdash_{\mathbb{C}} \overset{\circ}{\psi}_n(A) = B \text{ and } A \in \overset{\circ}{G}_n\}.$$

Let D_n^p be the closure of the domain of ψ_n^p , and let E_n^p be the closure of the interior of D_n^p —note that $D_n^p \setminus E_n^p$ is meagre since it is contained in the boundary of the closed set D_n^p . Let f_n^p be the maximal extension of ψ_n^p to a continuous function on a subset of E_n^p .

It must be the case that the domain of f_n^p is co-meagre in E_n^p , because if the domain of f_n^p is not co-meagre in E_n^p , then, because it is Borel, there must be some

¹Observe that the claim is made only for those $\overset{\circ}{g} \in \mathfrak{M}$ that are $\mathbb{P}(\mathfrak{F})$ names and not for $\mathbb{P}(\mathfrak{F}) \times \mathbb{Q}$ names in general.

open set $U \subset E_n^p$ such that the set of points in U to which ψ_n^p can be continuously extended is meagre in U . Since $p \Vdash \psi_n^p \subset \overset{\circ}{\psi}_n$ and because being a meagre Borel set is absolute, it must be that the set of points in U to which ψ_n can be continuously extended is also meagre in U . The reason is that the domain of ψ_n^p is dense in E_n^p , and so it follows that the domain of ψ_n^p is dense in U and, moreover, not being a point to which a function can be continuously extended is an absolute property. This contradicts the fact that G_n is co-meagre.

Now let

$$M' = \left(\bigcup_{n \in \omega \text{ and } p \in \mathbb{C}} D_n^p \setminus E_n^p \right) \cup \left(\bigcup_{n \in \omega \text{ and } p \in \mathbb{C}} E_n^p \setminus \text{domain}(f_n^p) \right)$$

and observe that M' is meagre. By the hypothesis on V it follows that the set $S =$

$$\left\{ C \in \mathcal{P}(A) : (\forall p \in \mathbb{C})(\forall k \in \omega) \psi_n^p(A) \text{ is not defined or } \psi_n^p(A) \triangle \Phi^*(A) \not\subseteq \bigcup_{i \leq k} \Phi^*(A_i) \right\}$$

is a second category set. Hence, S remains of second category after adding a Cohen real. Let N be a meagre set such that $G_n \supseteq \mathcal{P}(\omega) \setminus N$ for each $n \in \omega$, let $M = M' \cup N$ and choose $C \in S \setminus M$ in V' . But then there are some $p \in \mathbb{C}$ and $k \in \omega$ such that $\psi_n^p(A) \triangle \Phi^*(A) \subseteq \bigcup_{i \leq k} \Phi^*(A_i)$, which is a contradiction. \square

Theorem 3.1. *If V is a model of PFA and F is $\mathbb{P}(\kappa)$ generic over V , then, in $V[F]$, given any automorphism Φ of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ and disjoint infinite sets $\{A_n\}_{n \in \omega}$,*

$$(3.12) \quad (\exists A)(\exists \{\psi_n\}_{n \in \omega})(\forall n) A \supseteq^* A_n \text{ and}$$

$\psi_n : \mathcal{P}(A) \rightarrow \mathcal{P}(\omega)$ is a partial Borel function and

$$(\forall X \in \mathcal{P}(A))(\exists m, k \in \omega) \Phi^*(X) \triangle \psi_m(X) \subseteq \bigcup_{i \leq k} \Phi^*(A_i).$$

Proof. Let Φ be an automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ in $V[F]$ and suppose that $\{A_n\}_{n \in \omega}$ are infinite pairwise disjoint sets. Without loss of generality, $\emptyset \Vdash_{\mathbb{P}(\kappa)} \overset{\circ}{\Phi}$ is an automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ and (3.12) fails for $\{A_n\}_{n \in \omega}$ and $\overset{\circ}{\Phi}^*$ and $\Phi^*(A_i) \cap \Phi^*(A_j) = \emptyset$ unless $i = j$. Let G be a V -generic filter on a countably closed partial order, \mathbb{D} , which forces the existence of a \diamond -sequence $\{\mathfrak{M}_\xi\}_{\xi \in \omega_1}$.

As in [5], let $\mathfrak{F} = \{f_\eta\}_{\eta \in \omega_1}$ be a \leq_κ^* -increasing sequence in $V[G]$ such that

- for each $C \in V \cap \mathcal{P}(\omega) = V[G] \cap \mathcal{P}(\omega)$, the value of $\Phi^*(C)$ is decided by some f_η with respect to forcing by $\mathbb{P}(\kappa)$;
- the partial order $\mathbb{P}(\mathfrak{F})$ is ccc;
- forcing with $\mathbb{P}(\mathfrak{F})$ preserves dominating families in ${}^\omega\omega$;
- for each $A \in V$ such that $A \supseteq^* A_n$ for each n , forcing with $\mathbb{P}(\mathfrak{F})$ preserves that there do not exist Borel functions $\{\psi_n : n \in \omega\}$ such that for all but a meagre set of $C \in \mathcal{P}(A) \cap V$ there are $n \in \omega$ and $k \in \omega$ such that $\Phi^*(C) \triangle \psi_n(C) \subseteq \bigcup_{i \leq k} \Phi^*(A_i)$.

The first requirement is easily accomplished at successor steps by using $2^{\aleph_0} = \aleph_1$ and the fact that the forcing relation is not changed by modifying infinitely many coordinates of a condition by a finite amount. The second requirement is obtained by a standard \diamond argument, while the third is a consequence of Corollary 3.2. The last requirement is accomplished by appealing to the Omitting Type Theorem,

Lemma 2.1 on page 153 of [3], and applying it to the family of sentences $\sigma_C(x)$ parameterized by $C \in \mathcal{P}(A)$ which say that $x = (\{\psi_n : n \in \omega\}, M)$ where each $\psi_n : \mathcal{P}(A) \rightarrow \mathcal{P}(\omega)$ is a Borel function, M is a meagre F_σ subset of $\mathcal{P}(A)$, and either $C \in M$ or there is some $n \in \omega$ and $k \in \omega$ such that $\Phi^*(C) \triangle \psi_n(C) \subseteq \bigcup_{i \in k} \Phi^*(A_i)$. Lemma 3.6 asserts that Cohen forcing preserves that there is no x such that $\sigma_C(x)$ holds for each $C \in \mathcal{P}(A)$. The Omitting Type Theorem allows the conclusion that there is an oracle \mathfrak{N} such that forcing with any partial order that satisfies the \mathfrak{N} -cc preserves that there is no x such that $\sigma_C(x)$ holds for all $C \in \mathcal{P}(A)$; so it suffices to construct \mathfrak{F} to satisfy the \mathfrak{N} -cc. So, it may be assumed that $\mathfrak{N} = \{\mathfrak{M}_\eta\}_{\eta \in \omega_1}$.

Let $\{d_\xi\}_{\xi \in \omega_1}$ be a \leq^* -increasing, dominating family in ${}^\omega\omega$ in $V[G]$. It may, without loss of generality, be assumed that the sets A_n form a partition of ω and that $\bigcup_{n \in \omega} \Phi^*(A_n) = \omega$. Let $\Theta : \omega \rightarrow \omega \times \omega$ be a bijection in V such that the range of $\Theta \upharpoonright A_n$ is precisely $\{(n, m)\}_{m \in \omega}$. For $\alpha \in \beta \in \omega_1$ define $W_{\alpha, \beta} = \Theta^{-1}\{(n, m) : d_\alpha(n) \leq m < d_\beta(n)\}$ and let $W_{\alpha, \omega_1} = \Theta^{-1}\{(n, m) : d_\alpha(n) \leq m\}$. Observe that $W_{\alpha, \omega_1} \supseteq^* A_n$ for each $n \in \omega$ and $\alpha \in \omega_1$, while $W_{\alpha, \beta} \cap A_n$ is finite for each $n \in \omega$ and $\alpha \in \beta \in \omega_1$. Moreover, the sets $W_{\alpha, \beta}$ all belong to V .

Now, suppose that H is a $V[G]$ generic subset of $\mathbb{P}(\mathfrak{F})$. In $V[G][H]$, let $\{\dot{Y}_\eta\}_{\eta \in \omega_1}$ enumerate all possible names for reals with respect to hereditarily countable partial orders and let $\{\mathfrak{M}_\eta[H]\}_{\eta \in \omega_1}$ be an oracle given by \diamond (which exists since $\mathbb{P}(\mathfrak{F})$ has the countable chain condition, is of size \aleph_1 and \diamond holds in $V[G]$). It may also be assumed that $\{\dot{Y}_\xi\}_{\xi \in \zeta} \subseteq \mathfrak{M}_\zeta[H]$ for each $\zeta \in \omega_1$, since this can be accomplished by simply enlarging the oracle's guesses. Construct in $V[G][H]$ a sequence of pairs of ordinals $\{(\alpha_\eta, \beta_\eta)\}_{\eta \in \omega_1}$ and pairs $\{(V_\eta, W_\eta)\}_{\eta \in \omega_1}$ such that $\alpha_\eta \in \beta_\eta \in \alpha_\gamma$ if $\eta \in \gamma$ and $V_\eta \subseteq W_\eta \subseteq W_{\alpha_\eta, \beta_\eta}$. Furthermore, it will be required that, letting $\mathbb{S}_\zeta = \mathbb{S}(\{0_{W_\eta \setminus V_\eta} \cup 1_{V_\eta}\}_{\eta \in \zeta})$,

$$(3.13) \quad 1 \Vdash_{\mathbb{S}_\zeta} \text{ “ } \dot{\Phi}^*(V_\eta) \triangle \left(\dot{Y}_\mu \cap \dot{\Phi}^*(W_\eta) \right) \not\subseteq \bigcup_{i \in k} \dot{\Phi}^*(A_i) \text{ ”}$$

for each $\mu \leq \eta \in \zeta$ and any $k \in \omega$. In particular,

$$1 \Vdash_{\mathbb{S}_\zeta} \text{ “ } \dot{\Phi}^*(V_\eta) \triangle \left(\dot{Y}_\mu \cap \dot{\Phi}^*(W_\eta) \right) \neq^* \emptyset \text{ ”}$$

and so, if this induction can be carried out, then, using PFA, it is possible to argue just as in [5]. The main point is that the set H is generic for $\mathbb{P}(\mathfrak{F})$, which is the amoeba forcing of $\mathbb{P}(\kappa)$, and produces a condition f^* in $\mathbb{P}(\kappa)$ using Lemma 3.1. Moreover, f^* decides all that it is required to know about Φ^* . In other words, the argument² from [5] uses PFA to produce a subset of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ of size \aleph_1 which witnesses that Φ cannot be extended to a certain element of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ and f^* forces all the necessary values of Φ^* on this set of size \aleph_1 . Hence f^* contradicts that Φ is an automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ in spite of the fact that \emptyset forces this.

Now, suppose that the induction has been completed for all ordinals less than η . Let $\alpha_\eta \in \omega_1$ be so large that $\beta_\zeta \in \alpha_\eta$ for each $\zeta \in \eta$. Use Lemma 3.5, with \mathbb{Q} being a trivial partial order, to find a co-meagre set $\mathfrak{E} \subseteq \mathcal{P}(W_{\alpha_\eta, \omega_1})$ such that if $W \in \mathfrak{E}$, then all predense sets of \mathbb{S}_η belonging to $\mathfrak{M}_\eta[H]$ remain predense in

$$\mathbb{S}(\{0_{W_\zeta \setminus V_\zeta} \cup 1_{V_\zeta}\}_{\zeta \in \eta} \cup \{0_{W_\eta \setminus V_\eta} \cup 1_{V_\eta}\})$$

²The argument is reproduced in more detail, but in a slightly different context, in the proof of Theorem 4.1.

regardless of the choice of W_η or V_η provided that $V_\eta \subseteq W_\eta \subseteq W$ or $V_\eta \subseteq W_\eta \subseteq W_{\alpha_\eta, \omega_1} \setminus W$. Since dominating families in V remain dominating in $V[G][H]$, it follows that the reals of V remain of second category in $V[G][H]$; hence, it is possible to choose $W \subseteq W_{\alpha_\eta, \omega_1}$ belonging to $V \cap \mathfrak{E}$.

Let $W^0 = W$ and $W^1 = W_{\alpha_\eta, \omega_1} \setminus W$. For $i \in 2$, $p : D \rightarrow 2$ such that $p \in \mathbb{S}_\eta$, $k \in \omega$ and $\xi \in \eta$, define a partial function

$$\psi_{i,p,k,\xi} : \mathcal{P}(W^i) \rightarrow \mathcal{P} \left(\bigcup_{j=k}^{\infty} \Phi^*(A_j) \cap \Phi^*(W^i) \right)$$

by $\psi_{i,p,k,\xi}(C) = C'$ if and only if

$$(3.14) \quad (\forall n \in C') (\forall p' \in \mathbb{S}_\eta) \text{ if } p \cup p' \cup 0_{W^i \setminus (C \cup D)} \cup 1_{C \setminus D} \text{ is a function, then}$$

$$(\exists p'' \supseteq p \cup p') p'' \Vdash_{\mathbb{S}_\eta} "n \in \overset{\circ}{Y}_\xi" \text{ and } p'' \cup 0_{W^i \setminus (C \cup D)} \cup 1_{C \setminus D} \text{ is a function}$$

and

$$(3.15) \quad \left(\forall n \in \left(\bigcup_{j=k}^{\infty} \Phi^*(A_j) \cap \Phi^*(W^i) \right) \setminus C' \right) (\forall p' \in \mathbb{S}_\eta)$$

if $p \cup p' \cup 0_{W^i \setminus (C \cup D)} \cup 1_{C \setminus D}$ is a function, then

$$(\exists p'' \supseteq p \cup p') p'' \Vdash_{\mathbb{S}_\eta} "n \notin \overset{\circ}{Y}_\xi" \text{ and } p'' \cup 0_{W^i \setminus (C \cup D)} \cup 1_{C \setminus D} \text{ is a function.}$$

Define $\psi_{p,q,k,\xi,\zeta} : \mathcal{P}(W_{\alpha_\eta}) \rightarrow \mathcal{P}(\bigcup_{j=k}^{\infty} \Phi^*(A_j))$ by

$$\psi_{p,q,k,\xi,\zeta}(C) = \psi_{0,p,k,\xi}(C \cap W^0) \cup \psi_{1,q,k,\zeta}(C \cap W^1),$$

and observe that the functions $\psi_{p,q,k,\xi,\zeta}$ are all Borel. Hence there is some $C \in V$ such that

$$\psi_{p,q,k,\xi,\zeta}(C) \Delta \Phi^*(C) \not\subseteq \bigcup_{j \in k} \Phi^*(A_j)$$

for all p, q, k, ξ and ζ . It follows that there is some $i \in 2$ such that

$$(3.16) \quad \psi_{i,p,k,\xi}(C \cap W^i) \Delta \Phi^*(C \cap W^i) \not\subseteq \bigcup_{j \in k} \Phi^*(A_j) \cap \Phi^*(W^i)$$

for all p, k and ξ . Fix $i \in 2$ to be such that (3.16) is satisfied for all p, k and ξ .

Let $\{(p_j, \xi_j)\}_{j \in \omega}$ enumerate $\mathbb{S}_\eta \times \eta$ infinitely often and let D_j be the domain of p_j . For each j there is some $m_j \in \bigcup_{n=j}^{\infty} \Phi^*(A_n) \cap \Phi^*(W^i)$ such that if $\psi_{i,p_j,j,\xi_j}(C \cap W^i)$ is defined, then $m_j \in \psi_{i,p_j,j,\xi_j}(C \cap W^i) \Delta \Phi^*(C \cap W^i)$. In the case that $\psi_{i,p_j,j,\xi_j}(C \cap W^i)$ is defined, by appealing to (3.14) or (3.15), it is possible to find $p \in \mathbb{S}_\eta$ such that

$$(3.17) \quad p \supseteq p_j,$$

$$(3.18) \quad p \Vdash_{\mathbb{S}_\eta} "m_j \in \overset{\circ}{Y}_\xi \Delta \overset{\circ}{\Phi}^*(C \cap W^i) \setminus \bigcup_{n \in j} \overset{\circ}{\Phi}^*(A_n)",$$

$$(3.19) \quad p \cup 0_{W^i \setminus (C \cup D_j)} \cup 1_{C \setminus D_j} \text{ is a function.}$$

Furthermore, observe that if $\psi_{i,p_j,j,\xi_j}(C \cap W^i)$ is not defined, then there is some $m_j \in \bigcup_{k=j}^{\infty} \Phi^*(A_k) \cap \Phi^*(W^i)$ such that neither (3.14) nor (3.15) holds. To be precise, the failure of (3.14) means that there is some p_1 such that $p \cup p_1 \cup 0_{W^i \setminus (C \cup D_j)} \cup 1_{C \setminus D_j}$ is a function, but there is no $p'' \supseteq p \cup p_1$ such that $p'' \Vdash_{\mathbb{S}_\eta} "n \in \overset{\circ}{Y}_\xi"$ and

$p'' \cup 0_{W^i \setminus (C \cup D_j)} \cup 1_{C \setminus D_j}$ is a function. Similarly, the failure of (3.15) means that there is some p_2 such that $p \cup p_2 \cup 0_{W^i \setminus (C \cup D_j)} \cup 1_{C \setminus D_j}$ is a function, but there is no $p'' \supseteq p \cup p_2$ such that $p'' \Vdash_{\mathbb{S}_\eta}$ “ $n \notin \overset{\circ}{Y}_\xi$ ” and $p'' \cup 0_{W^i \setminus (C \cup D_j)} \cup 1_{C \setminus D_j}$ is a function. However, the fact that $\overset{\circ}{Y}_\xi \in \mathfrak{M}_\eta[H]$ implies that the sets of conditions deciding membership in $\overset{\circ}{Y}_\xi$ are dense in \mathbb{S}_η and hence in

$$\mathbb{S}(\{0_{W_\zeta \setminus V_\zeta} \cup 1_{V_\zeta}\}_{\zeta \in \eta} \cup \{0_{W_\eta \setminus V_\eta} \cup 1_{V_\eta}\})$$

as well. So there must be \bar{p}_1 and \bar{p}_2 such that

- $\bar{p}_1 \supseteq p \cup p_1$ and $\bar{p}_2 \supseteq p \cup p_2$;
- $\bar{p}_1 \Vdash_{\mathbb{S}_\eta}$ “ $n \notin \overset{\circ}{Y}_\xi$ ” and $\bar{p}_2 \Vdash_{\mathbb{S}_\eta}$ “ $n \in \overset{\circ}{Y}_\xi$ ”;
- $\bar{p}_1 \cup 0_{W^i \setminus (C \cup D_j)} \cup 1_{C \setminus D_j}$ and $\bar{p}_2 \cup 0_{W^i \setminus (C \cup D_j)} \cup 1_{C \setminus D_j}$ are both functions.

This makes it easy to once again find $p \in \mathbb{S}_\eta$ satisfying conditions (3.17), (3.18) and (3.19).

Since no unbounded reals are added by H it is possible to find $Z \in V$ such that $Z \supseteq \{m_j\}_{j \in \omega}$ and such that $Z \cap \Phi^*(A_n)$ is finite for each n . It follows that there is some β_η such that $\Phi^*(W_{\alpha_\eta, \beta_\eta}) \supseteq Z$. Let $W_\eta = W^i \cap W_{\alpha_\eta, \beta_\eta}$ and $V_\eta = C$.

To see that the pair (V_η, W_η) satisfies the required conditions, let $q \in \mathbb{S}_{\eta+1}$, $k \in \omega$ and $\xi \in \eta + 1$. It is possible to find $p : D \rightarrow 2$ such that $p \in \mathbb{S}_\eta$ and $q = p \cup 0_{W_\eta \setminus (V_\eta \cup D)} \cup 1_{V_\eta \setminus D}$. Let $j > k$ be such that $(p, \xi) = (p_j, \xi_j)$. By construction, there is some $p' \in \mathbb{S}_\eta$ such that conditions (3.17), (3.18) and (3.19) hold for the case j . In particular, by noting that $D = D_j$, $p' \cup p \cup 0_{W_\eta \setminus (V_\eta \cup D)} \cup 1_{V_\eta \setminus D} \supseteq p' \cup q$ is a function belonging to $\mathbb{S}_{\eta+1}$ and $p' \Vdash_{\mathbb{S}_\eta}$ “ $m_j \in \overset{\circ}{Y}_\xi \triangle \overset{\circ}{\Phi}^*(C \cap W^i) \setminus \bigcup_{n \in j} \overset{\circ}{\Phi}^*(A_n)$ ”. Hence, (3.13) holds, as required. \square

Corollary 3.3. *In the model $V[F]$, the ideal $\mathcal{T}(\Phi)$ is a dense P -ideal for every automorphism Φ of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$.*

Proof. Let $\{A_i\}_{i \in \omega}$ be disjoint sets in the ideal $\mathcal{T}(\Phi)$ and let this fact be witnessed by $g_i : A_i \rightarrow \omega$. From Theorem 3.1 it follows that there is A such that $A \supseteq^* A_n$ for each n and there are Borel functions ψ_n such that for all but a meagre set of $X \in \mathcal{P}(A)$ there are m and k such that $\Phi^*(X) \triangle \psi_m(X) \subseteq \Phi^*(\bigcup_{i \in k} A_i)$. Define

$$\psi_{i,j}(X) = \left(\bigcup_{m \in i} g_m(X \cap A_m) \right) \cup \left(\psi_j(X) \setminus \Phi^*\left(\bigcup_{m \in i} A_m\right) \right).$$

Lemma 2.1 applied to the Borel functions $\{\psi_{i,j}\}_{i,j \in \omega}$ yields that $A \in \mathcal{T}(\Phi)$.

To see that $\mathcal{T}(\Phi)$ is dense is immediate: given any set $X \subseteq \omega$, let $\{A_n\}_{n \in \omega}$ be disjoint infinite subsets of X and apply Theorem 3.1 to find A such that $A \supseteq^* A_n$ for each n and there are Borel functions ψ_n such that for all $Y \in \mathcal{P}(A)$ there are m and k such that

$$\Phi^*(Y) \triangle \psi_m(X) \subseteq \Phi^*\left(\bigcup_{i \in k} A_i\right).$$

Now choose an infinite $A' \subseteq A$ such that $A \cap A_n$ is finite for all n . It follows that $\Phi^*(Y) \triangle \psi_m(X)$ is finite for each $Y \subseteq A'$ and hence, from Lemma 2.1, it can be concluded that Φ is trivial on A' . \square

4. WHY THERE ARE FEW AUTOMORPHISMS

The next step is to establish the main result, which is that there are no more than κ^{\aleph_0} automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ in the model obtained by forcing with $\mathbb{P}(\kappa)$ over a model of PFA. In order to prove this, one is tempted to proceed by asserting that every automorphism is definable from the generic set. To establish this, suppose that \emptyset forces, with respect to $\mathbb{P}(\kappa)$, that for some automorphism Φ there is no Borel definition of Φ using the generic set as a parameter. Add a \diamond sequence to the initial model of PFA, and note that, since no new reals have been added, it is still forced, with respect to $\mathbb{P}(\kappa)$, that no Borel definition exists. The first difficulty is that it is no longer possible to iterate the two partial orders $\mathbb{P}(\mathfrak{F})$ and $\mathbb{S}(\mathfrak{S})$, since the first does add reals and so may add a Borel definition. Hence, the iteration must be replaced by a product. The induction involves simultaneously adding increasing conditions f_η and almost disjoint sets W_η such that $f_\eta \Vdash_{\mathbb{P}(\kappa)} "W_\eta \in \mathcal{T}(\Phi)"$. If the construction can be carried out, then, as usual, Φ cannot be extended. However, if the induction stops at stage ζ , then this means that for each set $W \in \mathcal{T}(\Phi)$ that is almost disjoint from each W_ξ for $\xi \in \zeta$, and for every $V \subseteq W$ there is some $\xi \in \zeta$ such that $\dot{Y}_\xi \cap g(W)$ is forced to be almost equal to $g(V)$ where $g : W \rightarrow \omega$ is a one-to-one function witnessing that $W \in \mathcal{T}(\Phi)$. Now the name \dot{Y}_ξ can be used to determine g . Using that $\mathcal{T}(\Phi)$ is a P -ideal it is possible to find a single $\xi \in \zeta$ that works for a dense set of W . Were it not for the fact that $g(W)$ needs to be known in order to use \dot{Y}_ξ to determine g , this would finish the argument. This difficulty needs to be dealt with by resorting to arguments from [6].

In the following theorem, the term “Borel” is used in the context of subsets of $\mathbb{P}(\kappa) \times \mathcal{P}(\omega)$; so its meaning should be explained. First, note that \mathbb{P} is a G_δ subset of ${}^\omega\omega$; so it inherits a Polish topology in a natural way. This yields a natural Polish topology on $\mathbb{P}^A \times \mathcal{P}(\omega)$ for any countable set A . A function B from $\mathbb{P}(\kappa) \times \mathcal{P}(\omega)$ to some Polish space will be said to be Borel if there is some countable set $A \subseteq \kappa$ and a Borel function B^* defined on $\mathbb{P}^A \times \mathcal{P}(\omega)$ such that $B(f) = B^*(f \upharpoonright A)$ for all $f \in \mathbb{P}(\kappa)$.

Theorem 4.1. *If V is a model of PFA and F is $\mathbb{P}(\kappa)$ generic over V and $\Phi \in V[F]$ is an automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$, then there is a Borel function B belonging to V and mapping $\mathbb{P}(\kappa) \times \mathcal{P}(\omega)$ to partial bijections of ω such that for all $W \in [\omega]^{\aleph_0}$ and $f \in F$ there is $\bar{W} \in [W]^{\aleph_0}$ and $\bar{f} \in F$ such that $\bar{f} \supseteq f$ and*

$$(\forall Z \subseteq \bar{W}) \Phi(Z) \equiv^* B(\bar{f}, \bar{W})(Z).$$

Moreover,³ $B(f, X) \subseteq B(f', X')$ provided that $B(f, X)$ and $B(f', X')$ are non-empty and $X \subseteq X'$ and $f \supseteq f'$.

Proof. Suppose that \emptyset forces, with respect to $\mathbb{P}(\kappa)$, that for some automorphism Φ there is no B satisfying the requirements of the theorem. As in the proof of Theorem 3.1, let $G \subseteq \mathbb{D}$ and add a \diamond sequence $\{\mathfrak{M}_\xi\}_{\xi \in \omega_1}$ to V , the initial model of PFA, and note that, since no new reals have been added, it is still forced, with respect to $\mathbb{P}(\kappa)$, that no B exists. For notational purposes only, fix an enumeration $\{\dot{Y}_\xi\}_{\xi \in \omega_1}$ of all possible countable names (with respect to any partial order whose

³Note that in the conclusion of Theorem 4.1, it is not claimed that $B(f, X)$ is invariant under finite changes to f .

elements are hereditarily countable) for subsets of ω . It may also be assumed that $\{\overset{\circ}{Y}_\xi\}_{\xi \in \zeta} \subseteq \mathfrak{M}_\zeta$ for each $\zeta \in \omega_1$, since this can be accomplished by simply enlarging \mathfrak{M}_ζ .

In $V[G]$ construct, if possible, a sequence $\{(f_\xi, W_\xi, V_\xi)\}_{\xi \in \omega_1}$ such that, letting $s_\xi = 0_{W_\xi \setminus V_\xi} \cup 1_{V_\xi}$ and $\mathbb{P}_\eta = \mathbb{P}(\{f_\xi\}_{\xi \in \eta}) \times \mathbb{S}(\{s_\xi\}_{\xi \in \eta})$, the following conditions hold:

- (1) For each η , $f_\eta \Vdash_{\mathbb{P}(\kappa)} "W_\eta \in \mathcal{T}(\overset{\circ}{\Phi})"$.
- (2) The partial functions $\{f_\xi\}_{\xi \in \omega_1}$ are increasing with respect to \leq_κ^* , while $\{W_\xi\}_{\xi \in \omega_1}$ form an almost-disjoint family.
- (3) For each $\eta \in \omega_1$ there are $A_\eta \subseteq \omega$ and $B_\eta \subseteq \omega$ such that

$$(4.1) \quad f_\eta \Vdash_{\mathbb{P}(\kappa)} "\overset{\circ}{\Phi}^*(W_\eta) = A_\eta \text{ and } \overset{\circ}{\Phi}^*(V_\eta) = B_\eta"$$

and for each $\zeta \in \eta$,

$$(4.2) \quad (\emptyset, \emptyset) \Vdash_{\mathbb{P}_\eta} "A_\eta \cap \overset{\circ}{Y}_\zeta \not\equiv^* B_\eta".$$

- (4) If $D \in \mathfrak{M}_\eta$ is a dense subset of \mathbb{P}_η , then D is a dense subset of \mathbb{P}_ζ for each $\zeta \geq \eta$.

Observe that since $\overset{\circ}{Y}_\zeta \in \mathfrak{M}_\eta$ if $\zeta \in \eta$, it follows that the dense sets required to decide membership in $\overset{\circ}{Y}_\zeta$ all belong to \mathfrak{M}_η . Since A_η and B_η belong to V , it follows that the dense subsets of \mathbb{P}_η required to guarantee condition (4.2) are subsets of \mathfrak{M}_η , but not necessarily elements of \mathfrak{M}_η . Nevertheless, since condition (4.2) only depends, in an absolute way, on the structure of the dense sets deciding membership in $\overset{\circ}{Y}_\zeta$, it follows from the last induction hypothesis that if \mathbb{P}_η satisfies condition (4.2), then so does \mathbb{P}_ζ for each $\zeta \geq \eta$.

If this construction succeeds, then it is possible to proceed as in [4] to prove that forcing with \mathbb{P}_{ω_1} adds a set to which the partial automorphism Φ cannot be extended. In particular, if $H_1 \times H_2$ is \mathbb{P}_{ω_1} generic, then, setting $X = \bigcup_{h \in H_2} h^{-1}\{1\}$, it follows that $X \cap W_\xi \equiv^* V_\xi$ for each $\xi \in \omega_1$, but in $V[G][H_1 \times H_2]$ for every $Y \subseteq \omega$ there is $\eta \in \omega_1$ such that for each $\beta \geq \eta$,

$$f_\beta \Vdash_{\mathbb{P}(\kappa)} "\overset{\circ}{\Phi}^*(W_\beta) = A_\beta \text{ and } \overset{\circ}{\Phi}^*(V_\beta) = B_\beta"$$

and $(\emptyset, \emptyset) \Vdash_{\mathbb{P}_\beta} "A_\beta \cap \overset{\circ}{Y} \not\equiv^* B_\beta"$ for each $\beta \geq \eta$. Just as in [4], it is possible to define a relation R on ω_1 by letting $R(\xi, \eta)$ hold if and only if

$$(B_\eta \cap (A_\xi \setminus B_\xi)) \cup (B_\xi \cap (A_\eta \setminus B_\eta)) \neq \emptyset.$$

It is easy to see that this is a semi-open relation—as defined by Abraham, Rubin and Shelah in [1] (or open as defined by Todorćević in [7])—and that, moreover, there is no $S \in [\omega_1]^{\aleph_1}$ such that $[S]^2 \cap R = \emptyset$. Hence, by the results of [1], there is a proper partial order \mathbb{K} that adds a set $S \in [\omega_1]^{\aleph_1}$ such that $[S]^2 \subseteq R$. This makes the fact that $\Phi \restriction V[G][H_1 \times H_2]$ cannot be extended to the set X absolute. The reason for *this* is that if there were a set Y such that $\Phi \restriction V[G][H_1 \times H_2]([X])$ could be defined to be $[Y]$, then it would be the case that $Y \cap \Phi^*(W_\xi) \equiv^* \Phi^*(V_\xi)$ for each $\xi \in S$. But then there would be an uncountable set $S' \subseteq S$, as well as $J \in \omega$, such that $Y \cap \Phi^*(W_\xi) \setminus J = \Phi^*(V_\xi) \setminus J$ for each $\xi \in S'$. It would follow that $\Phi^*(V_\xi) \setminus J \subseteq Y$ and that $(\Phi^*(W_\xi) \setminus \Phi^*(V_\xi)) \setminus J \subseteq \omega \setminus Y$ for each $\xi \in S'$. Choosing ξ and ζ in S' such that $\Phi^*(V_\xi) \cap J = \Phi^*(V_\zeta) \cap J$ and $\Phi^*(W_\xi) \cap J = \Phi^*(W_\zeta) \cap J$ would then yield the desired contradiction.

The iteration $\mathbb{D} * \mathbb{P}_{\omega_1} * \mathbb{K}$ is proper, and only \aleph_1 dense sets in it need be met in order to obtain S and the set X such that Φ cannot be extended to include $[X]$ in its domain. Using PFA, choose a filter $\Gamma \subseteq \mathbb{D} * \mathbb{P}_{\omega_1} * \mathbb{K}$ that meets all of the relevant dense sets and let f_Γ be the element of $\mathbb{P}(\kappa)$ obtained by intersecting Γ with $\mathbb{P}(\{f_\xi\}_{\xi \in \omega_1})$ —the first coordinate of \mathbb{P}_{ω_1} —and then appealing to Lemma 3.1. Let X_Γ be obtained by intersecting Γ with $\mathbb{S}(\{s_\xi\}_{\xi \in \omega_1})$, the second coordinate of \mathbb{P}_{ω_1} . The intersection of Γ with \mathbb{K} yields a set which witnesses that the fragment of Φ determined by f_Γ cannot be extended to include $[X_\Gamma]$ in its domain. It follows that $f_\Gamma \Vdash_{\mathbb{P}(\kappa)} \overset{\circ}{\Phi}$ “ Φ does not extend to $[X_\Gamma]$ ”. This contradicts that $\emptyset \Vdash_{\mathbb{P}(\kappa)} \overset{\circ}{\Phi}$ is an automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$.

Hence, suppose that the induction stops at stage η and that $\mathfrak{M} = \mathfrak{M}_\eta$ is the countable model presented by the \diamond sequence at that stage. Using Corollary 3.1 applied to \mathfrak{M} , it is possible to find $f^* \in \mathbb{P}(\kappa)$ such that $f_\zeta \leq_\kappa^* f^*$ for each $\zeta \in \eta$ and for any f' such that $f^* \leq^* f'$ where every $D \in \mathfrak{M}$ that is dense in \mathbb{P}_η remains dense in $\mathbb{P}(\{f'\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})$. Now apply Corollary 3.3 and the fact that $\mathbb{P}(\kappa)$ adds no new reals to find $\bar{f}_\eta \supseteq f^*$, \bar{W} and φ such that $\bar{W} \supseteq^* W_\zeta$ for each $\zeta \in \eta$ and $\bar{f}_\eta \Vdash_{\mathbb{P}(\kappa)} \varphi : \bar{W} \rightarrow \omega$ induces $\overset{\circ}{\Phi}$. Recall that the requirement $\bar{f}_\eta \supseteq f^*$ poses no problem because forcing with $\mathbb{P}(\kappa)$ is stable under finite modification of finitely many coordinates. Next, use Lemma 3.5 to find a partition $\omega \setminus \bar{W} = U^0 \cup U^1$ such that for any $a \in 2$ and any $W \subseteq U^a$ and any $V \subseteq W$, every $D \in \mathfrak{M}$ that is dense in $\mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})$ remains dense in $\mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta} \cup \{0_{W \setminus V} \cup 1_V\})$. Observe that the failure of the induction at stage η is absolute between models with the same reals; in particular, the construction fails in V at stage η if it fails in $V[G]$ at that stage. So the remaining argument takes place in V .

Now, for $a \in 2$, $f \in \mathbb{P}(\kappa)$, $q \in \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})$ and $\beta \in \eta$, define a relation $\rho_{f,q,\beta}^a \subseteq U^a \times \overset{\circ}{\Phi}^*(U^a)$ by letting $\rho_{f,q,\beta}^a(i, j)$ hold if and only if

$$(4.3) \quad (f, q \cup \{(i, 0)\}) \Vdash_{\mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})} \text{“} j \notin \overset{\circ}{Y}_\beta \text{”}$$

and

$$(4.4) \quad (f, q \cup \{(i, 1)\}) \Vdash_{\mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})} \text{“} j \in \overset{\circ}{Y}_\beta \text{”}.$$

It should be stressed here that the definition of $\rho_{f,q,\beta}^a$ depends only on those coordinates of $\mathbb{P}(\kappa)$ belonging to \mathfrak{M} because coordinates of f not belonging to \mathfrak{M} have no influence on membership in $\overset{\circ}{Y}_\beta$. This remark will be crucial later when claiming that the function B , yet to be defined, is Borel.

Claim 1. If $a \in 2$, $X \in [U^a]^{\aleph_0}$, $h : X \rightarrow \omega$ and $f \Vdash_{\mathbb{P}(\kappa)} \text{“} h \text{ induces } \overset{\circ}{\Phi} \text{”}$, then there are $\bar{f} \supseteq f$, q , and β such that $h \subseteq^* \rho_{\bar{f},q,\beta}^a$.

Proof. This is essentially the argument from (*15) on page 189 of [6]. □

Claim 2. If $\bar{f} \supseteq f$, then $\rho_{\bar{f},q,\beta}^a \supseteq \rho_{f,q,\beta}^a$ for any a , q and β .

Proof. Suppose that $\rho_{\bar{f},q,\beta}^a(i, j)$ holds but $\rho_{f,q,\beta}^a(i, j)$ fails. This means that either

$$(\bar{f}, q \cup \{(i, 0)\}) \not\Vdash_{\mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})} \text{“} j \notin \overset{\circ}{Y}_\beta \text{”}$$

or

$$(\bar{f}, q \cup \{(i, 1)\}) \not\Vdash_{\mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})} \text{“} j \in \overset{\circ}{Y}_\beta \text{”}.$$

So assume the first possibility, the argument in the other case being similar. However, the dense sets in \mathbb{P}_η , that decide whether or not $j \notin \mathring{Y}_\beta$, remain dense in $\mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})$ by the choice of \bar{f}_η . Hence there is some $(f', q') \in \mathbb{P}_\eta$ such that

$$(f' \cup \bar{f}, q' \cup q \cup \{(i, 0)\}) \in \mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})$$

and $(f', q') \Vdash_{\mathbb{P}_\eta} "j \in \mathring{Y}_\beta"$. But then

$$(f' \cup f, q' \cup q \cup \{(i, 0)\}) \in \mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})$$

because $f \subseteq \bar{f}$ and this contradicts that

$$(f, q \cup \{(i, 0)\}) \Vdash_{\mathbb{P}(\{\bar{f}_\eta\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})} "j \notin \mathring{Y}_\beta".$$

□

Claim 3. If $a \in 2$, $X \in [U^a]^{\aleph_0}$, $h : X \rightarrow \omega$ and $f \Vdash_{\mathbb{P}(\kappa)} "h \text{ induces } \mathring{\Phi}"$, then there exist \bar{f} , q and β such that $f \subseteq \bar{f}$ and $h \equiv^* \rho_{\bar{f}, q, \beta}^a \restriction X$.

Proof. Let $a \in 2$ be given and let $\{(q_n, \beta_n)\}_{n \in \omega}$ enumerate $\mathbb{S}(\{s_\zeta\}_{\zeta \in \eta}) \times \eta$ infinitely often. Let $f_0 = f$ and construct f_n inductively as follows. Given $f_m \in \mathbb{P}(\kappa)$ and a finite subset A_m of the support of f_m , choose an integer k_m large enough that for each $\alpha \in A_m$ there is some $k \in k_m$ such that $|[2^k, 2^{k+1}) \setminus \text{domain}(f_m(\alpha))| \geq m$. Let $\{g_i\}_{i=0}^L$ enumerate all $g \in \mathbb{P}\langle A_m, k_m \rangle$ such that $g \cup f_m \in \mathbb{P}(\kappa)$. Let $f_{m,0} = f_m$ and choose inductively, if possible, $f'_{m,i+1} \supseteq f_{m,i}$ such that $f'_{m,i+1} \langle A_m, k_m \rangle = f_{m,i} \langle A_m, k_m \rangle$ and $h \subseteq \rho_{f'_{m,i+1} \cup g_i, q_m, \beta_m}^a$. There are three possibilities to consider.

Case One. The first case to consider is that it is not possible to find $f'_{m,i+1}$ such that $f'_{m,i+1} \supseteq f_{m,i}$ and $f'_{m,i+1} \langle A_m, k_m \rangle = f_{m,i} \langle A_m, k_m \rangle$ and $h \subseteq \rho_{f'_{m,i+1} \cup g_i, q_m, \beta_m}^a$. In this case let $f_{m,i+1} = f_{m,i}$.

Case Two. If $f'_{m,i+1}$ is defined and $h \equiv^* \rho_{f'_{m,i+1} \cup g_i, q_m, \beta_m}^a \restriction X$, then let $\bar{f} = f'_{m,i+1} \cup g_i$, $q = q_m$ and $\beta = \beta_m$. This satisfies the conclusion of Claim 3.

Case Three. Otherwise, the range of $\rho_{f'_{m,i+1} \cup g_i, q_m, \beta_m}^a \setminus h$ contains an infinite set Z . The definition of $\rho_{f'_{m,i+1} \cup g_i, q_m, \beta_m}^a$ and Corollary 3.3 imply that there are $W_{m,i} \in [U^a]^{\aleph_0}$ and $f_{m,i+1} \supseteq f'_{m,i+1}$ such that

$$f_{m,i+1} \Vdash_{\mathbb{P}(\kappa)} "\mathring{\Phi}^*(W_{m,i}) \subseteq^* Z \text{ and } W_{m,i} \in \mathcal{T}(\mathring{\Phi})"$$

and $f_{m,i+1} \langle A_m, k_m \rangle = f_{m,i} \langle A_m, k_m \rangle$. Notice that the last condition is easily satisfied, since forcing with \mathbb{P} is stable under the finite modification of finitely many coordinates.

Now let $f_{m+1} = f_{m,L}$ and let $f_\omega = \bigcup_{m \in \omega} f_m$. If the sets A_m have been chosen wisely, the construction guarantees that $f_\omega \in \mathbb{P}(\kappa)$ and $f_{m,i} \leq_\kappa^* f_\omega$ for all m and i . Hence, from Corollary 3.3, it follows that there is some $W \subseteq \omega$ and $\bar{f} \supseteq f_\omega$ such that $W \supseteq X$ and $W \supseteq^* W_{m,i}$ for each relevant m and i and, furthermore, there is some one-to-one function $\bar{h} : W \rightarrow \omega$ such that $\bar{f} \Vdash_{\mathbb{P}(\kappa)} "\bar{h} \text{ induces } \mathring{\Phi} \text{ on } W"$. From Claim 1 it follows that there are $\bar{f} \supseteq \tilde{f}$ and m such that $\bar{h} \subseteq^* \rho_{\bar{f}, q_m, \beta_m}^a$. Since $X \subseteq W$, it follows that $h \subseteq^* \bar{h}$ and, hence, $h \subseteq^* \rho_{\bar{f}, q_m, \beta_m}^a$.

If $h \not\equiv^* \rho_{\bar{f}, q_m, \beta_m}^a \restriction X$, then it must be that there is some infinite Z contained in the range of $(\rho_{\bar{f}, q_m, \beta_m}^a \restriction X) \setminus h$. Let i be such that $g_i \supseteq \bar{f} \langle A_m, k_m \rangle$. It must be that Case 3 holds in the construction of $f_{m,i+1}$, since \bar{f} witnesses that Case 1 could not hold and, if Case 2 were in force, then the claim would already have been established. Therefore $W_{m,i} \subseteq^* W$. Using that $X \cap \bar{W}$ is finite, choose $j_1 \in X \setminus \text{domain}(q_m)$ and k belonging to the range of $\rho_{\bar{f}, q_m, \beta_m}^a \cap \bar{h} \restriction W_{m,i}$ such that $\rho_{f_{m,i+1} \cup g_i, q_m, \beta_m}^a(j_1, k)$ holds, $h(j_1) \neq k$ and $h(j_1) = \bar{h}(j_1)$. Let j_2 be such that $\bar{h}(j_2) = k$. Then $j_1 \neq j_2$ and $\rho_{\bar{f}, q_m, \beta_m}^a(j_1, k)$ holds from Claim 2 because $\bar{f} \supseteq f_\omega \cup g_i \supseteq f_{m,i+1} \cup g_i$. Moreover, $\rho_{\bar{f}, q_m, \beta_m}^a(j_2, k)$ also holds. Hence $(\bar{f}, q_m \cup \{(j_1, 0), (j_2, 1)\}) \Vdash_{\mathbb{P}(\{\bar{f}\}) \times \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})} "k \notin \overset{\circ}{Y}_\beta$ and $k \in \overset{\circ}{Y}_\beta"$. This contradiction means that Case 2 must have held at some point in the construction. \square

Claim 4. If $a \in 2$ and $f \in \mathbb{P}(\kappa)$, then there exist $f^* \supseteq f$, q and β such that for any $X \in [U^a]^{\aleph_0}$, $h : X \rightarrow \omega$ and $\bar{f} \supseteq f^*$ such that

$$\bar{f} \Vdash_{\mathbb{P}(\kappa)} "h \text{ induces } \Phi \text{ on } X",$$

there exist $\bar{f} \supseteq \tilde{f}$ such that $h \equiv^* \rho_{\bar{f}, q, \beta}^a \restriction X$.

Proof. Let $a \in 2$ be given. Let $\{(q_n, \beta_n)\}_{n \in \omega}$ enumerate $\mathbb{S}(\{s_\zeta\}_{\zeta \in \eta}) \times \eta$. Let $f_0 = f$ and construct f_n inductively as follows. Given $f_m \in \mathbb{P}(\kappa)$ and a finite subset A_m of the support of f_m , choose an integer k_m large enough that for each $\alpha \in A_m$ there is some $k \in k_m$ such that $|[2^k, 2^{k+1}) \setminus \text{domain}(f_m(\alpha))| \geq m$. Let $\{g_i\}_{i=0}^L$ enumerate all $g \in \mathbb{P} \langle A_m, k_m \rangle$ such that $g \cup f_m \in \mathbb{P}(\kappa)$. Let $f_{m,0} = f_m$ and choose inductively, if possible, $f_{m,i+1} \supseteq f_{m,i}$ such that $f_{m,i+1} \langle A_m, k_m \rangle = f_{m,i} \langle A_m, k_m \rangle$ and, moreover, there is some $h_{m,i} : X_{m,i} \rightarrow \omega$ such that $X_{m,i} \subseteq U^a$ and

$$f_{m,i+1} \Vdash_{\mathbb{P}(\kappa)} "h_{m,i} \text{ induces } \overset{\circ}{\Phi} \text{ on } X_{m,i}"$$

and $h_{m,i} \not\equiv^* \rho_{f', q_m, \beta_m}^a \restriction X_{m,i}$ for all $f' \supseteq f_{m,i+1} \cup g_i$. If this fails, then setting $f^* = f_{m,i} \cup g_i$, $q = q_m$ and $\beta = \beta_m$ satisfies the claim. Hence suppose that the construction never fails and let $f_{m+1} = f_{m,L}$ and define $f_\omega = \bigcup_{m \in \omega} f_m$. As in the proof of Claim 3, it is possible to find $\tilde{f} \supseteq f_\omega$ and $\tilde{h} : X \rightarrow \omega$ such that $U^a \supseteq X \supseteq^* X_{m,i}$ for all m and i and such that $\tilde{f} \Vdash_{\mathbb{P}(\kappa)} "h \text{ induces } \overset{\circ}{\Phi} \text{ on } X"$. Now use Claim 3 to find $\bar{f} \supseteq \tilde{f}$ and m such that $\tilde{h} \equiv^* \rho_{\bar{f}, q_m, \beta_m}^a$. Let i be such that $\bar{f} \langle A_m, k_m \rangle \subseteq g_i$ and note that $\bar{f} \cup g_i \supseteq f_{m,i+1} \cup g_i$ and $h_{m,i} \equiv^* \rho_{\bar{f} \cup g_i, q_m, \beta_m}^a \restriction X_{m,i}$. This contradicts the construction hypothesis. \square

Now, for any $(q_0, q_1) \in \mathbb{S}(\{s_\zeta\}_{\zeta \in \eta})^2$ and $(\beta_0, \beta_1) \in \eta^2$ define $B_{q_0, q_1, \beta_0, \beta_1}$ by

$$B_{q_0, q_1, \beta_0, \beta_1}(f, X) = \begin{cases} \varphi \restriction X & \text{if } X \subseteq \bar{W}, \\ \rho_{f, q_0, \beta_0} \restriction X & \text{if } X \subseteq U^0 \text{ and } \rho_{f, q_0, \beta_0} \restriction X \text{ is a function,} \\ \rho_{f, q_1, \beta_1} \restriction X & \text{if } X \subseteq U^1 \text{ and } \rho_{f, q_1, \beta_1} \restriction X \text{ is a function,} \\ \emptyset & \text{otherwise,} \end{cases}$$

where $f \in \mathbb{P}(\kappa)$ and $X \in [\omega]^{\aleph_0}$. Notice that $B_{q_0, q_1, \beta_0, \beta_1}$ is a Borel function. It will be shown that the set of conditions $f \in \mathbb{P}(\kappa)$ such that there are q_0, q_1, β_0 and β_1 satisfying

$$f \Vdash_{\mathbb{P}(\kappa)} "B_{q_0, q_1, \beta_0, \beta_1} \text{ satisfies the conclusion of Theorem 4.1}"$$

is dense.

To this end, let $f \in \mathbb{P}(\kappa)$ be arbitrary. Apply Claim 4 to f and $a = 0$ to find $f^0 \supseteq f$, q^0 and β^0 satisfying the conclusion of Claim 4. Then use Claim 4 again, but now applied to f^0 and $a = 1$ to find $f^1 \supseteq f^0$, q^1 and β^1 . Claim 4 together with the genericity of F over V guarantees that if $f^1 \in F$, then, in $V[F]$, for every $a \in 2$ and $X \in [U^a]^{\aleph_0}$ such that h induces Φ on X , there is some $\bar{f} \in F$ such that

$$h \equiv^* \rho_{\bar{f}, q^a, \beta^a} \restriction X$$

or, in other words, $B_{q^0, q^1, \beta^0, \beta^1}(\bar{f}, X)$ induces Φ on X . This amounts to the same as saying that f^1 forces that the Borel relation $B_{q^0, q^1, \beta^0, \beta^1}$ satisfies the conclusion of the theorem.

The final, “moreover”, clause is immediate from the definition of $B_{q^0, q^1, \beta^0, \beta^1}$. To see this, suppose that $B_{q^0, q^1, \beta^0, \beta^1}(f, X)$ and $B_{q^0, q^1, \beta^0, \beta^1}(f', X')$ are nonempty and $X \subseteq X'$ and $f \supseteq f'$. If $X' \subseteq \bar{W}$, then the result is clear; so let a be such that $X' \subseteq U_a$. Then Claim 2 implies that

$$(4.5) \quad \rho_{f, q^a, \beta^a} \supseteq \rho_{f', q^a, \beta^a}.$$

Since both $B_{q^0, q^1, \beta^0, \beta^1}(f, X)$ and $B_{q^0, q^1, \beta^0, \beta^1}(f', X')$ are nonempty, it follows that $\rho_{f, q^a, \beta^a} \restriction X$ and $\rho_{f', q^a, \beta^a} \restriction X'$ are both functions and, since $X \subseteq X'$, so is $\rho_{f', q^a, \beta^a} \restriction X$. From (4.5) it follows that

$$\rho_{f, q^a, \beta^a} \restriction X = \rho_{f', q^a, \beta^a} \restriction X$$

and hence $B_{q^0, q^1, \beta^0, \beta^1}(f, X) \subseteq B_{q^0, q^1, \beta^0, \beta^1}(f', X')$. \square

Corollary 4.1. *In $V[F]$ the number of automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ is between κ and κ^{\aleph_0} .*

Proof. From Lemma 3.3 it follows that it suffices to show that there are no more than κ^{\aleph_0} nontrivial automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$. If there are, then it follows from Theorem 4.1 that there are two automorphisms Φ and Ψ and there is a single Borel function B such that the following hold:

$$(4.6) \quad (\forall W \in [\omega]^{\aleph_0})(\forall f \in F)(\exists \bar{W} \in [W]^{\aleph_0})(\exists \bar{f} \in F)\bar{f} \supseteq f \text{ and} \\ \bar{f} \Vdash_{\mathbb{P}(\kappa)} “(\forall Z \subseteq \bar{W}) \overset{\circ}{\Phi}^*(Z) \equiv^* B(f, \bar{W})(Z)”,$$

$$(4.7) \quad (\forall W \in [\omega]^{\aleph_0})(\forall f \in F)(\exists \bar{W} \in [W]^{\aleph_0})(\exists \bar{f} \in F)\bar{f} \supseteq f \text{ and} \\ \bar{f} \Vdash_{\mathbb{P}(\kappa)} “(\forall Z \subseteq \bar{W}) \overset{\circ}{\Psi}^*(Z) \equiv^* B(f, \bar{W})(Z)”.$$

Using the fact that Ψ and Φ are forced to be different, it is possible to find $W \in [\omega]^{\aleph_0}$ and $f' \in F$ such that $f' \Vdash_{\mathbb{P}(\kappa)} “|\overset{\circ}{\Phi}^*(W) \cap \overset{\circ}{\Psi}^*(W)| < \aleph_0”$. From (4.6) it follows that there is $f_1 \in F$ such that $f_1 \supseteq f'$ and $X \subseteq W$ such that $f_1 \Vdash_{\mathbb{P}(\kappa)} “B(f_1, X) \text{ induces } \overset{\circ}{\Phi} \text{ on } X”$. From (4.7), it follows that there is some $f_2 \in F$ such that $f_2 \supseteq f_1$ and there is $\bar{X} \subseteq X$ such that $f_2 \Vdash_{\mathbb{P}(\kappa)} “B(f_2, \bar{X}) \text{ induces } \overset{\circ}{\Psi} \text{ on } \bar{X}”$. But now, applying the “moreover” clause of Theorem 4.1, it follows that $B(f_2, \bar{X}) \subseteq B(f_1, X)$, since both are nonempty. Hence

$$f_2 \Vdash_{\mathbb{P}} “\overset{\circ}{\Phi}^*(\bar{X}) \equiv^* B(f_1, X)(\bar{X}) \equiv^* B(f_2, \bar{X})(\bar{X}) \equiv \overset{\circ}{\Psi}^*(\bar{X})”,$$

contradicting that $\bar{X} \subseteq W$. \square

Corollary 4.2. *It is consistent that $2^{\aleph_0} = \aleph_2$, $2^{2^{\aleph_0}} = \aleph_4$ and the number of automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ is exactly \aleph_3 .*

Proof. Start with a model V of PFA where $2^{\aleph_2} = \aleph_4$ and then force with $\mathbb{P}(\aleph_3)$. Since $\aleph_3^{\aleph_0} = \aleph_3$ in the ground model, it follows from Corollary 4.2 that there are exactly \aleph_3 automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ in the generic extension. Since no new reals are added, the continuum does not change and because the forcing has size \aleph_3 , \aleph_4 is not collapsed. \square

5. COMMENTS

The automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ added by \mathbb{P} is trivial not only on a dense P -ideal, but on a dense P_{\aleph_1} -ideal. This raises the following question.

Question 5.1. In $V[F]$ is every automorphism trivial on a P_{\aleph_1} -ideal?

The next result is intended as motivation for the next question. Recall the definition of Φ_G in Definition 3.3 where G is a filter on \mathbb{P} .

Definition 5.1. Given a tower $\mathcal{F} = \{f_\xi\}_{\xi \in \kappa} \subseteq \mathbb{P}$, define \mathfrak{C}_ξ to be the set of all $a \in [\text{domain}(f_\xi)]^{<\aleph_0}$ such that $f_\xi \upharpoonright a$ is a cycle. Let

$$f_\xi^*(n) = \begin{cases} f_\xi(n) & \text{if } n \in a \in \mathfrak{C}_\xi \text{ and } |a| \text{ is odd,} \\ n & \text{otherwise.} \end{cases}$$

Proposition 5.1. *If a tower $\mathcal{F} = \{f_\xi\}_{\xi \in \kappa} \subseteq \mathbb{P}$ is obtained generically, then the tower of functions $\mathcal{F}^* = \{f_\xi^*\}_{\xi \in \kappa}$ as defined in Definition 5.1 is such that*

$$\Phi_{\mathcal{F}^*} \neq \Phi_{\mathcal{F}}^{n_0} g_1 \Phi_{\mathcal{F}}^{n_1} g_2 \Phi_{\mathcal{F}}^{n_2} \dots g_k \Phi_{\mathcal{F}}^{n_k}$$

for any choice of $\{n_i\}_{i=0}^k$ and almost permutations $\{g_i\}_{i=1}^k$. In other words, $\Phi_{\mathcal{F}^*}$ does not belong to the subgroup generated by $\Phi_{\mathcal{F}}$ and the trivial automorphisms.

Proof. Let $f \in \mathbb{P}$ and suppose that $\{n_i\}_{i=0}^k$ and $\{g_i\}_{i=1}^k$ are given. Choose $X \subseteq \omega$ such that $[2^x, 2^{x+1}) \setminus \text{domain}(f)$ has at least $(3k)^k \prod_{i=0}^k n_i$ elements for each $x \in X$ and, moreover, the set

$$\{[2^x, 2^{x+1}) \setminus \text{domain}(f) : x \in \omega \setminus X\}$$

is unbounded. For each $x \in X$ choose $j_x \in [2^x, 2^{x+1}) \setminus \text{domain}(f)$.

For any $j \in \omega$ let $\beta(j)$ be the set of all $g_i^{\epsilon_1} g_{i+1}^{\epsilon_2} \dots g_{i+m}^{\epsilon_m}(j)$ such that $1 \leq i$, $m \leq k-i$ and $(\epsilon_1, \epsilon_2, \dots, \epsilon_m) \in \{-1, 1\}^m$. For a set $a \subseteq \omega$, let $\bar{\beta}(a) = \bigcup_{j \in a} \beta(j)$. Now define an increasing sequence of functions f_x^n and integers j_x^n by induction, starting with $f_x^0 = \emptyset$ and $j_x^0 = j_x$. Let $N = \sum_{i \in k} |n_i| + k$. If $n \leq N$ and $n \notin \{\sum_{i \in m} |n_i| + m : m \leq k\}$, then there are three cases to consider. If $f_x^n(j_x^n)$ is defined, then let $f_x^n = f_x^{n+1}$ and $j_x^{n+1} = f_x^n(j_x^n)$. Otherwise, choose j_x^{n+1} in $[k_x, k_{x+1}) \setminus (\text{domain}(f) \cup \bar{\beta}(\{j_x^n\}_{i=0}^n))$. Then, if $\sum_{i \in m} |n_i| + m < n \leq n_m$ and $n_m > 0$, define $f_x^{n+1} = f_x^n \cup \{(j_x^n, j_x^{n+1})\}$ while, if $n_m < 0$, define $f_x^{n+1} = f_x^n \cup \{(j_x^{n+1}, j_x^n)\}$. If $n = \sum_{i \in m} |n_i| + m$ for some m , then define $f_x^{n+1} = f_x^n$ and $j_x^{n+1} = g_m(j_x^n)$.

Observe that f_x^N is one-to-one. Moreover, if m is such that $g_m(j_x^n) \neq j_x^n$ where $n = \sum_{i \in m} |n_i| + m$, then it is possible to extend f_x^N to two cycles, one containing j_x and the other containing

$$\Phi_{\mathcal{F}}^{n_0} g_1 \Phi_{\mathcal{F}}^{n_1} g_2 \Phi_{\mathcal{F}}^{n_2} \dots g_k \Phi_{\mathcal{F}}^{n_k}(j_x).$$

On the other hand, if $g_m(j_x^n) = j_x^n$ for all n such that $n = \sum_{i \in m} |n_i| + m$, then it is possible to extend f_x^N to an even permutation if $\sum_{i \in k} n_i \neq 0$ and to an odd cycle if $\sum_{i \in k} n_i = 0$. Let $f' = f \cup \bigcup_{x \in X} f_x^N$ and note that f' forces that

$$\Phi_{\mathcal{F}}^{n_0} g_1 \Phi_{\mathcal{F}}^{n_1} g_2 \Phi_{\mathcal{F}}^{n_2} \dots g_k \Phi_{\mathcal{F}}^{n_k} (\{j_x : x \in X\}) \not\equiv^* \Phi_{\mathcal{F}^*} (\{j_x : x \in X\}).$$

□

Question 5.2. Is it consistent that there is a nontrivial automorphism Ψ of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ yet every automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ is *algebraically* generated by the Ψ together with the trivial automorphisms? Proposition 5.1 shows that this is not so for the automorphisms and models discussed in this paper.

The gap between the two cardinals mentioned as possibilities in Corollary 4.1 points to the next question.

Question 5.3. Is it consistent that the cardinality of the automorphism group of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ has countable cofinality?

One might try to disprove this by assuming that the set of all automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ has cardinality \aleph_λ , where λ has countable cofinality, and then choosing a partition of the integers into infinite sets $\{A_n\}_{n \in \omega}$ and letting the set of all automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ be expressed as $\bigcup_{n \in \omega} \mathfrak{A}_n$ such that $|\mathfrak{A}_n| < \aleph_\lambda$. It is then possible to find automorphisms Φ_n of $\mathcal{P}(A_n)/[A_n]^{<\aleph_0}$ that disagree with every member of \mathfrak{A}_n . If these could be glued together somehow, a contradiction would soon be in hand. A solution of the “glueing” problem is likely to have other applications as well.

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